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LETTER TO THE EDITOR

Discussion of "The stress intensity factor for an external elliptical crack", *Int. J. Solids Structures*, Vol. 23, pp. 465–467 (1987)

In a recent paper[1] Fabrikant claimed that the stress intensity factor (K) formula given by Kassir and Sih[2] for an external elliptical crack in a three-dimensional solid due to an axial force P^{∞} is incorrect. He further showed that there is a considerable discrepancy between the correct K-formula he has derived and that obtained by Kassir and Sih. In this discussion we wish to show that Fabrikant's claim is unjustified and that there is confusion on his part about the parametric (Φ) and polar (ϕ) angles referred to the elliptical crack. In addition instead of solving the boundary value problem as Kassir and Sih did[2] we present here a simple method to obtain their K-formula which is correct.

Consider an elastic space weakened by an external elliptic crack in the plane $z = 0$ as shown in Fig. 1. Let a and b be the major and minor semi axes of the ellipse l_1. According to Kassir and Sih[2] the normal stress distribution (σ_{zz}) due to an axial force P^{∞} acting at infinity in the z-direction has the form

$$\sigma_{zz} = \frac{P^{\infty}}{2\pi ab\left[1 - \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2}\right]^{1/2}} \tag{1}$$

in which x, y and z are the Cartesian coordinates. The correct definition of K implies the normal approach to the crack border and is given by[2]

$$K = \lim_{r \to 0} \left[\sigma_{zz}(2r)^{1/2}\right] \tag{2}$$

in which r is the normal outward displacement of any point $P_1(x_1, y_1)$ on the ellipse l_1 along

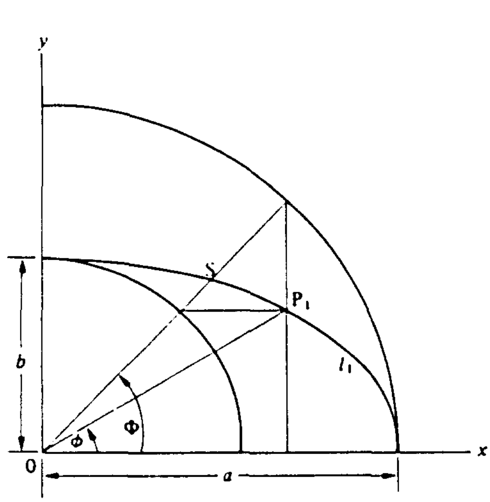

Fig. 1. Geometry of an elliptical crack showing the relationship between Φ and ϕ for point P_1 on the crack front.

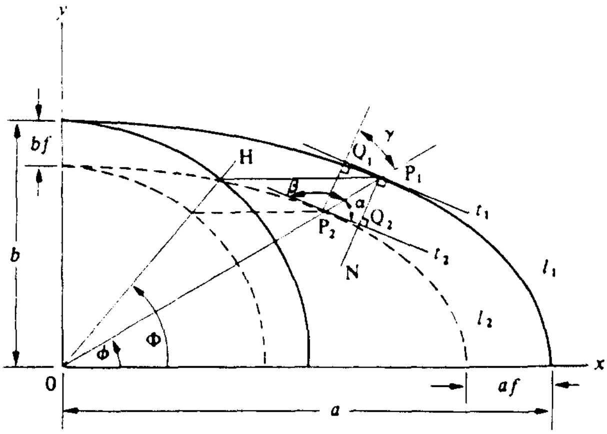

Fig. 2. Details of the geometries of ellipses l_1 and l_2 for the evaluation of eqn (2) for K.

the direction P_1N as shown in Fig. 2. The point $P_1(x_1, y_1)$ is defined by the parametric equations

$$x_1 = a\cos\Phi \tag{3a}$$

$$y_1 = b\sin\Phi. \tag{3b}$$

The parametric angle Φ and the polar angle ϕ of point $P_1(x_1, y_1)$ on ellipse l_1 are shown in Fig. 1. The polar radius of P_1 is given by

$$OP_1 = c(\Phi) = (a^2\cos^2\Phi + b^2\sin^2\Phi)^{1/2}. \tag{4}$$

If ellipse l_1 is reduced to l_2 by subtracting af from a and bf from b where f is very small, Fig. 2, then $P_1(x_1, y_1)$ on ellipse l_1 is also reduced to $P_2(x_2, y_2)$ along the polar radius OP_1. Thus, according to Green and Sneddon[3], P_1P_2 is equal to $fc(\Phi)$. If the polar radius of point $P_2(x_2, y_2)$ on ellipse l_2 is ρ, then

$$P_1P_2 = c(\Phi) - \rho \tag{5}$$

and the parametric equations for P_2 become

$$x_2 = (1-f)a\cos\Phi \tag{6a}$$

$$y_2 = (1-f)b\sin\Phi. \tag{6b}$$

The straight line t_2 tangential to point $P_2(x_2, y_2)$ on ellipse l_2 is described by

$$\frac{x_2x}{(1-f)^2a^2} + \frac{y_2y}{(1-f)^2b^2} - 1 = 0 \tag{7}$$

and the gradient m_2 of line t_2 at P_2 is

$$m_2 = -(b/a)\cot\Phi. \tag{8}$$

Similarly, a straight line t_1 tangential to P_1 on ellipse l_1 can be described by

$$\frac{x_1x}{a^2} + \frac{y_1y}{b^2} - 1 = 0 \tag{9}$$

so that its gradient m_1 is given by

$$m_1 = -(b/a) \cot \Phi = m_2. \tag{10}$$

Thus lines t_1 and t_2 are parallel to each other and so are ellipses l_1 and l_2. When ellipse l_1 is reduced to ellipse l_2, the normal outward displacement r of point $P_1(x_1, y_1)$ along P_1N is simply the perpendicular distance P_1Q_2 to line t_2 which is tangential to P_2 on ellipse l_2. Consider the right-angled triangle $P_1Q_2P_2$. Let α be the angle between the polar radius OP_1 and line t_2, and β be the complementary angle. It is easy to show that

$$r = P_1Q_2 = P_1P_2 \sin \alpha = P_1P_2 \sin \beta = [c(\Phi)-\rho] \sin \beta \tag{11}$$

using eqn (5) for P_1P_2. From analytic geometry, Fig. 2, the gradient m_2 at $P_2(x_2, y_2)$ on ellipse l_2 can be written as

$$m_2 = \tan (\phi+\beta) \tag{12}$$

and

$$\tan \phi = (b/a) \tan \Phi. \tag{13}$$

Now, from eqns (4), (12) and (13), we have

$$\sin \beta = \frac{ab}{c(\Phi)[a^2 \sin^2 \Phi + b^2 \cos^2 \Phi]^{1/2}}. \tag{14}$$

Also, for point $P_2(x_2, y_2)$ the Cartesian coordinates (x_2, y_2) are given by

$$x_2 = \rho \cos \phi = \frac{\rho a \cos \Phi}{c(\Phi)} \tag{15a}$$

$$y_2 = \rho \sin \phi = \frac{\rho b \sin \Phi}{c(\Phi)}. \tag{15b}$$

Substituting eqns (15) into eqn (1) yields

$$\sigma_{zz} = \frac{c(\Phi)P^{\varepsilon}}{2\pi ab[c^2(\Phi)-\rho^2]^{1/2}}. \tag{16}$$

The K-formula can now be obtained from eqn (2) using eqns (11), (14) and (16). In the limit when $r \to 0$, $c(\Phi) \to \rho$ so that

$$K(\Phi) = \frac{P^{\varepsilon}}{2\pi(ab)^{1/2}[a^2 \sin^2 \Phi + b^2 \cos^2 \Phi]^{1/4}}. \tag{17}$$

This result for K is identical to that given by Kassir and Sih[2] and it refers to the stress intensity factor at point P_1 on ellipse l_1 being defined by the parametric angle Φ. Although P_1 can also be defined by the polar angle ϕ, eqn (7) only gives K as a function Φ. To calculate K at P_1 in terms of ϕ, we can use the relation between ϕ and Φ from eqn (13) in eqn (17). Thus, we have

$$K(\phi) = \frac{P^{\varepsilon}}{2\pi(ab)^{1/2}}\left[\frac{a^2 \sin^2 \phi + b^2 \cos^2 \phi}{a^4 \sin^2 \phi + b^4 \cos^2 \phi}\right]^{1/4} \tag{18}$$

which is the so-called correct K-formula derived by Fabrikant[1].

It is obvious, therefore, that the correct stress intensity factor at P_1 can be given by either or both eqns (17) and (18) depending on whether the parametric angle Φ or the polar angle ϕ is used. In stress intensity factor handbooks, e.g. Tada *et al.*[4], it is often easy to misinterpret $K(\Phi)$ as that stress intensity factor for point S (and not point P_1), Fig. 1, which is the intersection point of OS defined by the parametric angle Φ and the elliptical crack front. To avoid this confusion Fabrikant's $K(\phi)$ of eqn (18) to calculate K for a point such as P_1 on the elliptical crack front is preferred since it can be unambiguously defined by the polar angle ϕ and the polar radius OP_1.

Now returning to Fabrikant's paper in which he asserted that eqn (17) of Kassir and Sih is incorrect, it seems that he has got mixed up with the two angles Φ and ϕ and has wrongly interpreted Φ as ϕ in the K-equation (17). Equation (5) in his paper is therefore wrong and it corresponds to the incorrect definition of K in which

$$K = \lim_{\rho \to c(\phi)} \{\sigma_{zz} 2[c(\phi) - \rho]^{1/2}\}. \tag{19}$$

Had he realized the angle in Kassir and Sih's K-formula of eqn (17) is in fact the parametric and not the polar angle he would have easily derived the "correct" K-formula of eqn (18) in terms of the polar angle. Both $K(\Phi)$ and $K(\phi)$ are correct as shown in this discussion and they refer to the *same* point on the elliptical crack front. Consequently, Figs 2 and 3 in Fabrikant's paper which purport to show the discrepancy between the "incorrect" and "correct" K-formulae are meaningless and misleading.

There is nothing wrong with Kassir and Sih's formula of eqn (17) but care must be taken that Φ is a parametric angle and not the polar angle as is assumed in Fabrikant's paper.

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AUTHOR'S CLOSURE

It was strange to read a discussion being much longer than the original paper. All the main objections raised by Zhang and Mai were responded to in my closure related to the remarks by Kassir and Sih [1], and will not be repeated here. The reader is addressed to the above-mentioned closure. Here I present some specific notes related to the discussion by Zhang and Mai.

(1) The real confusion is not in my paper, but in the book of Kassir and Sih (and some other books which I do not name here taking into consideration present experience) where ϕ on each drawing is clearly indicated as the polar angle, while now they claim that the same parameter ϕ in their formulae stands for a parametric angle. I repeat once again that my paper was sent to both Kassir and Sih two years ago, and if the situation was clear to them at that time, they could have responded with an explanation but they did not.